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TURBULENCE IN MULTIPHASE FLOW

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Abstract—We establish in this paper the foundations of a two-field turbulent flow model that includes two turbulent fields. The case of dispersed particles in an incompressible carrier fluid is treated here, but the very presence of these two fields allows for the generalization of the model to the instability-induced turbulent mixing of two materials. This model describes both cases of turbulent mass diffusion and small drag regime, "wave-like" interpenetration of the two components. It also includes the damping of the turbulence due to the presence of the particles. In addition, a theoretical derivation of the drag-induced decay of the large-scale turbulence kinetic energy is proposed as another mechanism specific to turbulent multiphase flow.

Key Words: turbulence, modeling, two-field mixing

1. INTRODUCTION

Turbulence in two-phase flow has been a subject of interest for many years. However, only low concentration suspensions are relatively well understood; in such a flow, the turbulence of the mainstream is not affected by the dispersed phase. Those cases when the turbulence in the surrounding fluid is affected are much more difficult.

Experimental work has shown that the presence of solid particles or liquid droplets modifies the turbulence structure. On the theoretical side, very little is known. Turbulence models, as in Elgobashi & Abou-Arab (1983) and Margolin (1977), predict the effect of particles on the turbulence intensity. However, these earlier studies investigated the limiting case of a small volume fraction for the dispersed phase. This simplified the model, requiring equations for the carrier fluid only. The difficulty with any two-phase turbulent flow model resides in the interaction between the dispersed particles and the carrier fluid (e.g. Daly & Harlow 1978). Two different approaches have been used to describe the behavior of particle-laden turbulent flow. Using the dynamics of single particles, some authors, such as Gouesbet *et al.* (1984), obtained turbulence intensity correlations, and Margolin (1977) calculated the diffusion coefficient for particles in a turbulent flow. The other approach is to use averaged field equations (see Nigmatulin 1979) to develop a turbulence model, as in Elgobashi & Abou-Arab (1983).

We investigate here the case of a larger volume fraction and develop the foundations of a model accounting for two different turbulent fields. This will allow for its generalization to the description of such phenomenon as the instability-induced turbulent interpenetration of two materials (see Besnard & Harlow 1987). Also described by this model is the drag-induced decay of large-scale turbulence kinetic energy, and the limiting cases of turbulent mass diffusion, and a small-drag, "wave-like" interpenetration regime. The more classical features of return to isotropy, and damping of the turbulence kinetic energy due to the presence of particles in the surrounding fluid are also accounted for by the model.

We start here with two-phase fluid equations, and, as it is necessary to proceed in a stepwise manner, we restrict ourselves to the case of rigid particles in an incompressible fluid. In these

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equations, the field variables have been defined by appropriate averaging (Nigmatulin 1979):

$$\epsilon_1 + \epsilon_2 = 1, \tag{1}$$

$$\frac{\partial \epsilon_1}{\partial t} + \frac{\partial \epsilon_1 U_{1j}}{\partial X_j} = 0, \tag{2}$$

$$\frac{\partial \epsilon_2}{\partial t} + \frac{\partial \epsilon_2 U_{2j}}{\partial X_j} = 0, \tag{3}$$

$$\frac{\partial \epsilon_1 U_{1i}}{\partial t} + \frac{\partial \epsilon_1 U_{1i} U_{1j}}{\partial X_j} = -\frac{\epsilon_1}{\rho_1}\frac{\partial P}{\partial X_i} + \frac{K}{\rho_1}(U_{2i} - U_{1i}) \tag{4}$$

$$\frac{\partial \epsilon_2 U_{2i}}{\partial t} + \frac{\partial \epsilon_2 U_{2i} U_{2j}}{\partial X_j} = -\frac{\epsilon_2}{\rho_2}\frac{\partial P}{\partial X_i} + \frac{K}{\rho_2}(U_{1i} - U_{2i}), \tag{5}$$

where ϵ_1 and ϵ_2 are the volume fractions of the particles and the incompressible fluid, respectively, ρ_1 and ρ_2 are their microscopic densities, U_{1i} and U_{2i} are their velocities and K is the coupling function describing the interaction between the two fields. The pressure gradient $\partial P/\partial X_i$ is obtained after a careful integration of the microscopic gradients in the control volume (see Nigmatulin 1979). At a given point $\mathbf{x}$, it represents the external forces that accelerate solid particles or fluid elements located at $\mathbf{x}$. In the case where the particle size is not small compared to the length scale of interest, the above averaging is not possible, and pressure would have to be defined within and outside the particles. Terms describing viscous effects have been omitted, due to the difficulty of defining the viscosity of the dispersed phase, and there are no collision terms. In order for our model to approach the usual single-phase turbulence transport equations, it is necessary to include the effect of viscosity as $\epsilon_1 \to 0$. Away from that limit, however, the effects of momentum exchange between the fields predominate, and the viscous stresses are negligible. We are therefore restricted to a certain range for ϵ_1. The lower limit, $\epsilon_{1\,\mathrm{min}}$, depends on the level of viscosity; the upper limit, $\epsilon_{1\,\mathrm{max}}$, occurs when the collision rate between particles becomes important.

It is assumed (see Daly & Harlow 1978) that K can be modeled as the product of $\epsilon_1\epsilon_2$ and a function that depends only on the local properties of the two fields, e.g.

$$K = C_D \epsilon_1 \epsilon_2 \rho_2 |U_2 - U_1| \equiv \epsilon_1 \epsilon_2 C.$$

We have stressed that [1]–[5] are already averaged. This means that the size of the particles is assumed to be much smaller than the size of the control volume over which the averaging has been done. As a consequence, this also means that any length scale that we deal with in our derivation is also assumed to be much larger than the size of the control volume. This suggests that only large-scale turbulence (compared to the size of particles) can be described here.

The usual approach for describing turbulence transport in two-field flow is to separate each of the variables of the system [1]–[5] into a mean part and a fluctuating part (see Lumley 1978) and to follow the same procedure used in one-fluid turbulent transport by Daly & Harlow (1970). For a single field, the results are transport equations for the Reynolds stress tensor R_{ij} and for the energy-decay-rate tensor D_{ij}.

For two-field turbulent transport, we want to define entities for each of the fluids analogous to the Reynolds stress tensor. More precisely, we want to derive a conservation equation for the total mean momentum including the turbulent flux of momentum. One approach is to generalize for the two-field case models that have been developed for a single fluid. This approach is especially attractive when the first field is sparsely distributed in the background fluid, since the second fluid then determines the behavior of the mixture. The procedure is to introduce a mean and fluctuating part for the volume fraction and velocity of each field:

$$\left.\begin{aligned} U_{1i} &= \bar{U}_{1i} + U'_{1i}, \\ \epsilon_1 &= \bar{\epsilon}_1 + \epsilon'_1, \\ U_{2i} &= \bar{U}_{2i} + U'_{2i}, \\ \epsilon_2 &= \bar{\epsilon}_2 + \epsilon'_2. \end{aligned}\right\} \tag{6}$$

The use of the space-averaged [1]–[5] introduces questions regarding the length scale for two-field turbulence. Our question pertains to the relationship between particle size, interparticle distance and the spectrum of scales in that part of the flow that is considered to be turbulent. Within the context of two-field flow, the mean variations of the field variables take place over distances that are large compared to the particle size and the interparticle spacing. We assume this also to be true for all relevant scales of turbulence, despite the recognition that turbulence scales comparable to particle size and spacing inevitably exist whenever there is relative motion between the fields. Precise resolution of the question raised by this discrepancy in our assumptions can only be settled with much additional investigation. For now, we confine our considerations of scales to those parts of the spectrum that are sufficiently large for the two-field equations to be valid.

This is made possible by the basic assumption that the length scale of the turbulence is much greater than the length scale of the ensemble averaging volume, and much smaller than the characteristic length of our problem, i.e. of the mean flow. This splitting into mean and fluctuating parts can be applied to any variable of interest. There is, however, a major conceptual difficulty with the choice of the volume fraction and velocity as the primary variables. Because the velocity is not a transportable quantity, it is impossible to define a conservation equation for the mean total momentum, $\rho_1\bar{\epsilon}_1\bar{U}_{1i} + \rho_2\bar{\epsilon}_2\bar{U}_{2i}$. The alternative approach, is to choose $m_{1i} = \epsilon_1 U_{1i}$ and $m_{2i} = \epsilon_2 U_{2i}$ as primary variables. This gives, for two-phase flow, the same type of model "B", that Favre (1965) described for a single compressible flow.

Having chosen m_1 and m_2 as the primary variables, we still require a definition for the average velocities of the two fields. Following Besnard & Harlow (1985) we define $\tilde{U}_{1i}$ and $\tilde{U}_{2i}$ as $\bar{m}_{1i} = \bar{\epsilon}_1\tilde{U}_{1i}$ and $\bar{m}_{2i} = \bar{\epsilon}_2\tilde{U}_{2i}$. In the limiting case of the one-field model, we recover the usual definition for the unweighted average velocity. The objective of the present work is to develop a turbulence model that accounts for the full dynamic behavior of the two fields and the interaction between them.

We derive our equations in section 2, and show that it is convenient to introduce a hierarchy of models, more and more complex, in order to describe whichever level of precision we need. However, we will restrict ourselves to the lowest order model. We show the necessity for a length scale for the turbulence, particularly in order to close the pressure–velocity correlations. We then show that this turbulence length scale can be eliminated from the equations through the introduction of an energy-decay-rate tensor, which is very similar to the tensor obtained in the one-field case. The different closures are derived in section 3.

Experiments show that in many types of interpenetrating two-field flows, there is a steady state for which one possible interpretation is to consider the ensemble average of the fluctuating flow to be the mean-flow part, with the rest described as turbulence. In some circumstances, Needham & Nerkin (1983) showed that the collisional pressure P_s, coming from collisions between particles, can stabilize both the linear and the non-linear perturbations to the flow. We show in section 4 that the Reynolds tensor can play the same role as P_s, and exhibit the source term for the turbulence equations corresponding to the instabilities of the mean flow developing into turbulence.

In section 5, we present an analysis of this model, and demonstrate that it can adequately describe the diffusive behavior of the volume fraction in the case of strong coupling, as described by Lumley (1975, 1978), but also that it exhibits a "wave-like" limit that corresponds to a very weak coupling between the two fields. We also show that the presence of the first fluid induces a strong effect on the turbulence of the background second fluid through the coupling terms.

2. DERIVATION OF THE EQUATIONS

2.1. Turbulent energy equations

We rewrite the system [1]–[5] as follows:

$$\epsilon_1 + \epsilon_2 = 1, \qquad [7]$$

$$\frac{\partial \epsilon_1}{\partial t} + \frac{\partial m_{1k}}{\partial X_k} = 0, \qquad [8]$$

$$\frac{\partial \epsilon_2}{\partial t} + \frac{\partial m_{2k}}{\partial X_k} = 0, \qquad [9]$$

$$\frac{\partial m_{1i}}{\partial t}+\frac{\partial}{\partial X_k}\left(\frac{m_{1i}m_{1k}}{\epsilon_1}\right)=-\frac{\epsilon_1}{\rho_1}\frac{\partial P}{\partial X_i}+\frac{C}{\rho_1}(\epsilon_1 m_{2i}-\epsilon_2 m_{1i}) \quad [10]$$

and

$$\frac{\partial m_{2i}}{\partial t}+\frac{\partial}{\partial X_k}\left(\frac{m_{2i}m_{2k}}{\epsilon_2}\right)=-\frac{\epsilon_2}{\rho_2}\frac{\partial P}{\partial X_i}+\frac{C}{\rho_2}(\epsilon_2 m_{1i}-\epsilon_1 m_{2i}). \quad [11]$$

Using the decompositions $m_{1i}=\overline{m_{1i}}+m'_{1i}$ and $m_{2i}=\overline{m_{2i}}+m'_{2i}$, [7]–[9] become

$$\bar{\epsilon}_1+\bar{\epsilon}_2=1, \quad [12]$$

$$\frac{\partial\bar{\epsilon}_1}{\partial t}+\frac{\partial\overline{m_{1k}}}{\partial X_k}=0 \quad [13]$$

and

$$\frac{\partial\bar{\epsilon}_2}{\partial t}+\frac{\partial\overline{m_{2k}}}{\partial X_k}=0. \quad [14]$$

The momentum equations are not so straightforward. For the first field, we obtain

$$\frac{\partial\overline{m_{1i}}}{\partial t}+\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1i}}\,\overline{m_{1k}}}{\bar{\epsilon}_1}\right)=-\overline{\frac{\epsilon_1}{\rho_1}\frac{\partial P}{\partial X_i}}-\frac{\partial F_{1ik}}{\partial X_k}+\overline{\frac{C}{\rho_1}(\epsilon_1 m_{2i}-\epsilon_2 m_{1i})}. \quad [15]$$

In this equation F_{1ik} is the Reynolds stress tensor and its expression is

$$F_{1ik}=\overline{\left(\frac{m_{1i}m_{1k}}{\epsilon_1}\right)}-\frac{\overline{m_{1i}}\,\overline{m_{1k}}}{\bar{\epsilon}_1}. \quad [16]$$

The fluctuating part C' of the coupling coefficient C should be calculated as a function of the fluxes and the volume fractions of the two fields. However, its precise expression depends on whether we consider Stokes drag, or flow separation drag, or any other type of interaction between the two fields. For the sake of simplicity, we neglect the fluctuating part of C.

Our next task is to derive a transport equation for the tensors F_{1ij} and F_{2ij}. Let us first define $\psi_1=1/\epsilon_1$. Using [2]–[4], we obtain

$$\frac{\partial}{\partial t}(m_{1i}m_{1j}\psi_1)=-\frac{\partial}{\partial X_k}(m_{1i}m_{1j}m_{1k}\psi_1^2)+\left(m_{1i}\frac{\partial P}{\partial X_j}+m_{1j}\frac{\partial P}{\partial X_i}\right)+\frac{C}{\rho_1}(m_{1j}m_{2i}+m_{1i}m_{2j}-2\epsilon_2 m_{1i}m_{1j}\psi_1). \quad [17]$$

Due to the definition of F_{1ij}, the only term where ψ_1 appears explicitly is in the gradient of $m_{1i}m_{1j}m_{1k}\psi_1^2$. Notice that $\psi_1=\Sigma_n\epsilon_2^n$. We then can derive a hierarchy of approximate equations, based on this expansion. Also notice that $\epsilon'_1=-\epsilon'_2$, and that $|\epsilon'_1|<\bar{\epsilon}_1$ and $|\epsilon'_2|<\bar{\epsilon}_2$ (i.e. ϵ'_1 and ϵ'_2 are bounded quantities). If $h^2=\max(\bar{\epsilon}_1^2,\bar{\epsilon}_2^2)$, we have $\bar{\psi}_1=1/\bar{\epsilon}_1+O(h^2)$. However, for moderate volume fraction variance (i.e. $\overline{\epsilon_1'^2}<\bar{\epsilon}_1^2$), the above decomposition of ψ_1 should be rearranged in the form $\psi_1=1/\bar{\epsilon}_1\,\Sigma_n(\epsilon'_1/\bar{\epsilon}_1)^n$, this formal series being convergent after the averaging procedure. Taking an average of [17], we have

$$\overline{m_{1i}m_{1j}m_{1k}\psi_1^2}=\overline{m_{1i}m_{1j}m_{1k}\left[\frac{1}{\bar{\epsilon}_1}\sum_n\left(\frac{\epsilon'_1}{\bar{\epsilon}_1}\right)^n\right]^2},$$

of which we keep the first term of the series, $\overline{m_{1i}m_{1j}m_{1k}}/\bar{\epsilon}_1^2$. This is equivalent to discarding all correlations $\overline{\epsilon_1'^a A'_1\ldots A'_r}$ when compared to $\bar{\epsilon}_1^a\overline{A'_1\cdots A'_r}$. Although this lowest order approximation seems rather limiting, it is enough to exhibit the main properties of this model, based on a careful analysis of the coupling terms between the two fields.

However, we must emphasize that the fluctuations in m contain the fluctuations in ϵ; thus, even when only the lowest order in the expansion is retained, the theory is very different from what we would have obtained if we had neglected volume fraction correlations altogether. Keeping higher order correlations in ϵ is crucial for describing instability-induced turbulence [i.e. Rayleigh–Taylor, in Besnard & Harlow (1987)].

After averaging [17], we obtain the correlations $T_{ij} = \overline{m'_{1i} m'_{2i}}$ and $U_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$. In order to adequately describe the interaction between the two fields, we need an equation for T_{ij}. Using the fact that the correlation $\overline{m'_{1i} \partial m'_{ij}/\partial X_j}$ is of first order with respect to ϵ'_1 (Besnard & Harlow 1985) this equation for T_{ij} can be simplified, and the resulting equations for F_{1ij}, U_{ij} and F_{2ij}, to the lowest order, are:

$$\frac{\partial F_{1ij}}{\partial t} + \frac{\overline{m_{1k}}}{\bar{\epsilon}_1}\frac{\partial F_{1ij}}{\partial X_k} + F_{1ij}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1k}}}{\bar{\epsilon}_1}\right) + F_{1ik}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1j}}}{\bar{\epsilon}_1}\right) + F_{1jk}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1i}}}{\bar{\epsilon}_1}\right)$$
$$= -\frac{\partial}{\partial X_k}\overline{(m'_{1i} m'_{1j} m'_{1k}/\bar{\epsilon}_1^2)} - \overline{\left(m'_{1j}\frac{\partial P'}{\partial X_i} + m'_{1i}\frac{\partial P'}{\partial X_j}\right)} + \frac{2\bar{C}}{\rho_1}(U_{ij} - \bar{\epsilon}_2 F_{1ij}), \qquad [18]$$

$$\frac{\partial F_{2ij}}{\partial t} + \frac{\overline{m_{2k}}}{\bar{\epsilon}_2}\frac{\partial F_{2ij}}{\partial X_k} + F_{2ij}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{2k}}}{\bar{\epsilon}_2}\right) + F_{2ik}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{2j}}}{\bar{\epsilon}_2}\right) + F_{2jk}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{2i}}}{\bar{\epsilon}_2}\right)$$
$$-\frac{\partial}{\partial X_k}\overline{(m'_{2i} m'_{2j} m'_{2k}/\bar{\epsilon}_2^2)} - \overline{\left(m'_{2j}\frac{\partial P'}{\partial X_i} + m'_{2i}\frac{\partial P'}{\partial X_j}\right)} + \frac{2\bar{C}}{\rho_2}(U_{ij} - \bar{\epsilon}_1 F_{2ij}), \qquad [19]$$

and

$$\frac{\partial T_{ij}}{\partial t} + T_{ij}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1k}}}{\bar{\epsilon}_1} + \frac{\overline{m_{2k}}}{\bar{\epsilon}_2}\right) + T_{kj}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1i}}}{\bar{\epsilon}_1}\right) + T_{ik}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{2j}}}{\bar{\epsilon}_2}\right)$$
$$+ \frac{\overline{m_{1k}}}{\bar{\epsilon}_1}\overline{m'_{2j}\frac{\partial m'_{1i}}{\partial X_k}} + \frac{\overline{m_{2k}}}{\bar{\epsilon}_2}\overline{m'_{1i}\frac{\partial m'_{2j}}{\partial X_k}} + \overline{m'_{2j}\frac{\partial}{\partial X_k}\left(\frac{m'_{1i} m'_{1k}}{\bar{\epsilon}_1}\right)}$$
$$+ \overline{m'_{1i}\frac{\partial}{\partial X_k}\left(\frac{m'_{2j} m'_{2k}}{\bar{\epsilon}_2}\right)}$$
$$= -\frac{\bar{\epsilon}_1}{\rho_1}\overline{m'_{2j}\frac{\partial P'}{\partial X_i}} - \frac{\bar{\epsilon}_2}{\rho_2}\overline{m'_{1i}\frac{\partial P'}{\partial X_j}} + \frac{\bar{C}}{\rho_1}\bar{\epsilon}_1(\bar{\epsilon}_2 F_{1ij} - T_{ij}) + \frac{\bar{C}}{\rho_2}\bar{\epsilon}_2(\bar{\epsilon}_1 F_{2ij} - T_{ij}). \qquad [20]$$

The l.h.s. of [18] can be rewritten as

$$\frac{\partial F_{1ij}}{\partial t} + (\mathscr{L}_u F_1)_{ij} = [\omega, F_1]_{ij}$$

where $(\mathscr{L}_u F_1)$ is the Lie derivative of F_i, in the direction of the fluid, ω is the mean vorticity and $[\omega, F_1]$ the commutator of ω and F_1. The two first terms represent the time variation of the tensor F_{1ij} along the motion of the fluid. The last term is a rearrangement term among the F_1 components, because it has zero trace. The source term for turbulence due to the rotation and stretching of the mean fluid elements is imbedded into the covariant derivative $\mathscr{L}_u F_1$. The additional terms are other correlations that require modeling. Equations [18]–[20] are coupled to the mean flow equations, consisting of [12]–[14] and the mean momentum equations

$$\frac{\partial \overline{m_{1i}}}{\partial t} + \frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1i} m_{1k}}}{\bar{\epsilon}_1}\right) + \frac{\partial F_{1ik}}{\partial X_k} = -\frac{\bar{\epsilon}_1}{\rho_1}\frac{\partial \bar{P}}{\partial X_i} + \frac{\bar{C}}{\rho_1}(\bar{\epsilon}_1 \overline{m_{2i}} - \bar{\epsilon}_2 \overline{m_{1i}}) \qquad [21]$$

and

$$\frac{\partial \overline{m_{2i}}}{\partial t} + \frac{\partial}{\partial X_k}\left(\frac{\overline{m_{2i} m_{2k}}}{\bar{\epsilon}_2}\right) + \frac{\partial F_{2ik}}{\partial X_k} = -\frac{\bar{\epsilon}_2}{\rho_2}\frac{\partial \bar{P}}{\partial X_i} + \frac{\bar{C}}{\rho_2}(\bar{\epsilon}_2 \bar{m}_{1i} - \bar{\epsilon}_1 \overline{m_{2i}}). \qquad [22]$$

2.2. Dissipation of turbulent energy

We have restricted ourselves to the scales of the turbulence spectrum that are sufficiently large for the two-field equations to be valid. Even within the limit of this model, there is a matter of considerable importance that must be addressed, regarding the question of dissipation in the equations for F_{1ij} and F_{2ij}. The normal derivation of the turbulence energy decay rate ignores the cascade process from structures of wavelength λ_n to structures of smaller wavelength λ_{n+1}, $\forall n$. The rate of decay of turbulence $S_{n,n+1}$ is the source rate for turbulence of order $n+1$, F_{n+1}.

Then we write

$$\frac{\mathrm{d}F_n}{\mathrm{d}t} = S_{n-1,n} - S_{n,n+1}.$$

When an equilibrium is reached, the decay rate $S_{n,n+1}$, for any n, is exactly balanced by the ultimate dissipation of the smallest scale eddies into heat as a result of molecular dissipation. The source rate $S_{0,1}$ is the turbulence rate provided through the mean-flow instabilities.

This allows us to calculate the decay rate of turbulent energy into heat using the cascade process from large-scale structures to smaller ones. This differs from the usual derivation of the transport equations for the Reynolds stress-tensor components from the Navier–Stokes equations, which leads to the introduction of the energy-decay-rate tensor arising from the viscous dissipation term, as in Daly & Harlow (1970).

In the single-fluid case, it is shown in Besnard & Harlow (1985) that the decay of large-scale turbulence to smaller-scale turbulence can be obtained from a careful reinterpretation of the transport terms of the Reynolds stress-tensor equation. In the two-field case this alternate method of derivation becomes essential in the absence of viscosity, and in any case is preferable in that it shows the crucial identification of decay as coming from the energy containing eddies as a result of cascade, rather than from dissipation directly into heat.

Noticing that the total momentum is convected with the mean velocity

$$\tilde{U}_k = \left(\frac{\rho_1 \overline{m_{1k}} + \rho_2 \overline{m_{2k}}}{\bar{\epsilon}_1 \rho_1 + \bar{\epsilon}_2 \rho_2}\right),$$

we only keep that convective term for the correlation T.

In order to demonstrate this, let us rewrite the transport equations for F_{1ij}, F_{2ij} and T_{ij}, neglecting multiple correlations:

$$\frac{\partial F_{1ij}}{\partial t} + U_{1k}\frac{\partial F_{1ij}}{\partial X_k} + F_{1ij}\frac{\partial U_{1k}}{\partial X_k} + F_{1ik}\frac{\partial U_{1j}}{\partial X_k} + F_{1jk}\frac{\partial U_{1i}}{\partial X_k}$$

$$= 2\frac{C}{\rho_1}(U_{ij} - \bar{\epsilon}_2 F_{1ij}) + \frac{1}{\rho_1}\overline{P'\left(\frac{\partial m'_{1i}}{\partial X_j} + \frac{\partial m'_{1j}}{\partial X_i}\right)}, \qquad [23]$$

$$\frac{\partial F_{2ij}}{\partial t} + U_{2k}\frac{\partial F_{2ij}}{\partial X_k} + F_{2ij}\frac{\partial U_{2k}}{\partial X_k} + F_{2ik}\frac{\partial U_{2j}}{\partial X_k} + F_{2jk}\frac{\partial U_{2i}}{\partial X_k}$$

$$= 2\frac{C}{\rho_2}(U_{ij} - \bar{\epsilon}_1 F_{2ij}) + \frac{1}{\rho_2}\overline{P'\left(\frac{\partial m'_{2i}}{\partial X_j} + \frac{\partial m'_{2j}}{\partial X_i}\right)} \qquad [24]$$

and

$$\frac{\partial T_{ij}}{\partial t} + \tilde{U}_k\frac{\partial T_{ij}}{\partial X_k} + T_{ij}\frac{\partial}{\partial X_k}(U_{1k} + U_{2k}) + T_{kj}\frac{\partial U_{1i}}{\partial X_k} + T_{ik}\frac{\partial U_{2j}}{\partial X_k}$$

$$= \frac{C}{\rho_1}\bar{\epsilon}_2(\bar{\epsilon}_1 F_{2ij} - T_{ij}) + \frac{C}{\rho_2}\bar{\epsilon}_1(\bar{\epsilon}_2 F_{1ij} - T_{ij}) - \frac{\bar{\epsilon}_1}{\rho_1}\frac{\overline{P'\partial m'_{1j}}}{\partial X_i} - \frac{\bar{\epsilon}_2}{\rho_2}\frac{\overline{P'\partial m'_{1i}}}{\partial X_j}, \qquad [25]$$

in which over bars have been omitted from the mean velocities occurring in these equations.

The pressure correlations in [23]–[25] are crucial because they lead to a simple rearrangement among the different components of the F_{1ij} and F_{2ij} tensors, and give some insight into obtaining the deviator part of F_{1ij} and F_{2ij}.

Contracting [23] for the first fluid and [24] for the second one, and adding the two resulting equations, we obtain

$$2\frac{\partial}{\partial t}(\rho_1 F_{1ii} + \rho_2 F_{2ii}) = \overline{P'\frac{\partial m'_{1i}}{\partial X_i}} + \overline{P'\frac{\partial m'_{2i}}{\partial X_i}},$$

where spatial derivatives vanish because we have neglected inhomogeneities here.

We also have from the mass equations,

$$\overline{P'\frac{\partial m'_{1i}}{\partial X_i}} = -\overline{P'\frac{\partial \epsilon'_1}{\partial t}}$$

and

$$\overline{P'\frac{\partial m'_{2i}}{\partial X_i}} = -\overline{P'\frac{\partial \epsilon'_2}{\partial t}}.$$

From $\epsilon'_1 + \epsilon'_2 = 0$, it follows that the total turbulent kinetic energy per unit volume remains unaffected by the pressure correlations. We interpret this result to mean that only a rearrangement among the various components of the Reynolds stress tensor can take place here. Moreover, within our approximation, we also notice that the contraction $\overline{P'\partial m'_i/\partial X_i}$ must vanish to lowest order in an expansion in powers of ϵ'_1. Thus, we choose the simplest possible model for these correlations, i.e. a linear model, and write

$$\overline{P'\frac{\partial m'_{1j}}{\partial X_i} + P'\frac{\partial m'_{1i}}{\partial X_j}} = A\rho_1\left(F_{1kk}\frac{\delta_{ij}}{3} - F_{1ij}\right) \qquad [26]$$

and

$$\overline{P'\frac{\partial m'_{2j}}{\partial X_i}} + \overline{P'\frac{\partial m'_{2i}}{\partial X_j}} = A\rho_2\left(F_{2kk}\frac{\delta_{ij}}{3} - F_{2ij}\right). \qquad [27]$$

A few comments must be made here. First, the expressions in [26] and [27] are only a lowest order approximation for these terms. Some ideas about their first-order approximation are presented in Besnard & Harlow (1985). Secondly, we notice that the effects of coupling between the two fields are taken into account in the term $C/\rho_1(U_{ij} - \bar{\epsilon}_2 F_{1ij})$ and its counterpart in [24]. We speculate that A must be independent of C for cases in which the turbulence level is sufficiently large, because we attribute the component rearrangement principally to displacement interactions between particles and the incompressible fluid. We also speculate that the stronger the turbulence the faster must be the return to isotropy for both fields. For our present goal, it is sufficient to write $A = a^{-1}\omega$, where ω^{-1} is a dimensionless perturbation parameter.

Let us now define $q_1 = F_{1kk}/2$, $q_2 = F_{2kk}/2$, $q_{12} = T_{kk}/2 = U_{kk}/2$. We need approximations for F_{1ij} and F_{2ij}, which we take similar to the low-order approximation usually introduced for the one-field Reynolds stress, namely

$$\left.\begin{aligned} F_{1ij} &= \tfrac{2}{3}q_1\delta_{ij} + \frac{e_{1ij}}{\omega}, \\ F_{2ij} &= \tfrac{2}{3}q_2\delta_{ij} + \frac{e_{2ij}}{\omega}, \\ T_{ij} &= \tfrac{2}{3}q_{12}\delta_{ij} + T^*_{ij}, \end{aligned}\right\} \qquad [28]$$

where

$$T^*_{ij} = T_{ij} - \left(\frac{\delta_{ij}}{3}\right)T_{kk}.$$

Notice that these expressions lead to a simplified transport equation, analogous to the k–ϵ model for one-fluid turbulence (see Besnard & Harlow 1985).

Assuming a nearly isotropic case, i.e. that ω is very large, we can insert [28] in [23]–[25] and keep only the lowest order terms in powers of ω^{-1}. We then obtain the following system of equations for q_1, q_2 and q_{12}:

$$\frac{\partial q_1}{\partial t} + U_{1k}\frac{\partial q_1}{\partial X_k} + \tfrac{5}{3}q_1\frac{\partial U_{1k}}{\partial X_k} = 2\frac{C}{\rho_1}(q_{12} - \bar{\epsilon}_2 q_1), \tag{29}$$

$$\frac{\partial q_2}{\partial t} + U_{2k}\frac{\partial q_2}{\partial X_k} + \tfrac{5}{3}q_2\frac{\partial U_{2k}}{\partial X_k} = 2\frac{C}{\rho_2}(q_{12} - \bar{\epsilon}_1 q_2) \tag{30}$$

and

$$\frac{\partial q_{12}}{\partial t} + \tilde{U}_k\frac{\partial q_{12}}{\partial X_k} + \tfrac{4}{3}q_{12}\frac{\partial}{\partial X_k}(U_{1k} + U_{2k}) = \frac{C}{\rho_1}\bar{\epsilon}_2(\bar{\epsilon}_1 q_1 - q_{12}) + \frac{C}{\rho_2}\bar{\epsilon}_1(\bar{\epsilon}_2 q_2 - q_{12}). \tag{31}$$

In order to obtain the equations for e_{1ij}, e_{2ij} and T^*_{ij}, we multiply [29]–[31] by δ_{ij} and subtract them from the equations obtained by putting [28] into the uncontracted [23]–[25]. We then obtain

$$e_{1ij} = -\tfrac{2}{3}a\left[q_1\left(\frac{\partial U_{1i}}{\partial X_j} + \frac{\partial U_{1j}}{\partial X_i}\right) - \tfrac{2}{3}q_1\delta_{ij}\frac{\partial U_{1k}}{\partial X_k}\right] \tag{32}$$

and, similarly,

$$e_{2ii} = -\tfrac{2}{3}a\left[q_2\left(\frac{\partial U_{2i}}{\partial X_j} + \frac{\partial U_{2j}}{\partial X_i}\right) - \tfrac{2}{3}q_2\delta_{ij}\frac{\partial U_{2k}}{\partial X_k}\right].$$

T^*_{ij} is deduced from [28] and [31]. Expression [32] are crude approximations for e_{1ij} and e_{2ij}. Nevertheless, they have the usual form of the approximation for the deviator part of the Reynolds stress tensor in the single-fluid case. With this preparation we can now proceed with the derivation of the decay terms for the F_{1ij} and F_{2ij} equations.

This model is only valid for turbulence length scales larger than the length scale over which the microscopic fluid equations have been averaged. We introduce the distinction between the large-scale part of the turbulence we wish to describe, and the small-scale part which eventually dissipates into heat. The decay rate between large-scale and small-scale turbulence is balanced by the ultimate dissipation into heat, as discussed in the introduction to this section.

The procedure here is to divide the momentum m_{1i} (and m_{2i}) into three parts (e.g. $m_{1i} = \overline{m_{1i}} + m^{l}_{1i} + m^{s}_{1i}$), which are assumed to be uncorelated, put these expressions into the momentum equations for both fields and derive equations for the Reynolds stress tensors F^{l}_{1ij}, F^{s}_{1ij}, F^{l}_{2ij} and F^{s}_{2ij}. It is sufficient here to work with approximate equations, and we start with the following set of equations for F^{l}_{1ij} and F^{s}_{1ij}, corresponding to the spectrum $(\bar{U}_{1i}, U^{l}_{1i}, U^{s}_{1i})$ for the velocity component U_{1i} $(U_{1i} = \bar{U}_{1i} + U^{l}_{1i} + U^{s}_{1i})$:

$$\frac{\partial F^{l}_{1ij}}{\partial t} + \bar{U}_{1k}\frac{\partial F^{l}_{1ij}}{\partial X_k} = -F^{l}_{1kj}\frac{\partial \bar{U}_{1i}}{\partial X_k} - F^{l}_{1ik}\frac{\partial \bar{U}_{1j}}{\partial X_k} - F^{l}_{1ij}\frac{\partial \bar{U}_{1k}}{\partial X_k} + F^{s}_{1jk}\frac{\partial U^{l}_{1i}}{\partial X_k} + F^{s}_{1ik}\frac{\partial U^{l}_{1j}}{\partial X_k} + F^{s}_{1ij}\frac{\partial U^{l}_{1k}}{\partial X_k} \tag{33}$$

and

$$\frac{\partial F^{s}_{1ij}}{\partial t} + \bar{U}_{1k}\frac{\partial F^{s}_{1ij}}{\partial X_k} = -F^{s}_{1kj}\frac{\partial U^{l}_{1i}}{\partial X_k} - F^{s}_{1ik}\frac{\partial U^{l}_{1j}}{\partial X_k} - F^{s}_{1ij}\frac{\partial U^{l}_{1k}}{\partial X_k} - F^{s}_{1jk}\frac{\partial \bar{U}_{1i}}{\partial X_k} - F^{s}_{1ik}\frac{\partial \bar{U}_{1j}}{\partial X_k} - F^{s}_{1ij}\frac{\partial \bar{U}_{1k}}{\partial X_k}. \tag{34}$$

Adding these equations, we recover the equation for $F_{1ij} = F^{l}_{1ij} + F^{s}_{1ij}$, as expected.

We put [28] into [33], using [32], and define $a_1 = \tfrac{2}{3}aq^{s}_{1}$ and $a_2 = \tfrac{2}{3}aq^{s}_{2}$ to obtain

$$\frac{\partial F^{l}_{1ij}}{\partial t} + \bar{U}_{1k}\frac{\partial F_{1ij}}{\partial X_k} + F^{l}_{1kj}\frac{\partial \bar{U}_{1i}}{\partial X_k} + F^{l}_{1ik}\frac{\partial \bar{U}_{1j}}{\partial X_k} + F^{l}_{1ij}\frac{\partial \bar{U}_{1k}}{\partial X_k} = \bar{H}_{1ij}, \tag{35}$$

where

$$H_{1ij}=\tfrac{2}{3}q_1^s\left[\left(\frac{\partial U^1_{1i}}{\partial X_j}+\frac{\partial U^1_{1j}}{\partial X_i}\right)+\delta_{ij}\frac{\partial U^1_{1k}}{\partial X_k}\right]-\frac{a_1}{\omega}\left\{2\frac{\partial U^1_{1i}}{\partial X_k}\frac{\partial U^1_{1j}}{\partial X_k}\right.$$
$$+\left(\frac{\partial U^1_{1i}}{\partial X_j}+\frac{\partial U^1_{1j}}{\partial X_i}\right)\frac{\partial U^1_{1k}}{\partial X_k}+\frac{\partial U^1_{1k}}{\partial X_j}\frac{\partial U^1_{1i}}{\partial X_k}+\frac{\partial U^l_{1k}}{\partial X_i}\frac{\partial U^l_{1j}}{\partial X_k}$$
$$\left.-\frac{2}{3}\frac{\partial U^1_{1k}}{\partial X_k}\left[\left(\frac{\partial U^1_{1i}}{\partial X_j}+\frac{\partial U^1_{1j}}{\partial X_i}\right)+\delta_{ij}\frac{\partial U^1_{1k}}{\partial X_k}\right]\right\}. \qquad [36]$$

The large-scale velocities average to zero, so that returning to our usual notation

$$\bar{H}_{1ij}=-\frac{a_1}{\omega}\left\{2\overline{\frac{\partial U'_{1i}}{\partial X_k}\frac{\partial U'_{1j}}{\partial X_k}}+\overline{\left(\frac{\partial U'_{1i}}{\partial X_j}+\frac{\partial U'_{1j}}{\partial X_i}\right)\frac{\partial U'_{1k}}{\partial X_k}}+\overline{\frac{\partial U'_{1k}}{\partial X_j}\frac{\partial U'_{1i}}{\partial X_k}}\right.$$
$$\left.+\overline{\frac{\partial U'_{1k}}{\partial X_j}\frac{\partial U'_{1j}}{\partial X_k}}-\overline{\frac{2}{3}\frac{\partial U'_{1k}}{\partial X_k}\left[\left(\frac{\partial U'_{1i}}{\partial X_j}+\frac{\partial U'_{1j}}{\partial X_i}\right)+\delta_{ij}\frac{\partial U'_{1k}}{\partial x_k}\right]}\right\}. \qquad [37]$$

A few comments must be made here. In the limiting case of a single fluid, we obtain

$$\bar{H}_{1ij}=-\frac{a_1}{\omega}\left[2\overline{\frac{\partial U'_{1i}}{\partial X_k}\frac{\partial U'_{1j}}{\partial X_k}}+\frac{\partial}{\partial X_k}\overline{\left(U'_{1i}\frac{\partial U'_{1k}}{\partial X_j}+U'_{1j}\frac{\partial U'_{1k}}{\partial X_i}\right)}\right]. \qquad [38]$$

The expression for $\bar{H}_{2ij}$ is obtained in similar manner.

The second term on the r.h.s. of [38] vanishes for homogeneous turbulence. This implies that this term is not part of the energy-decay-rate tensor itself, but merely describes non-homogeneity effects. Thus, we obtain the result described in Daly & Harlow (1970) for the energy-decay-rate tensor

$$D_{ij}=\frac{1}{2}\overline{\frac{\partial U'_{1i}}{\partial X_k}\frac{\partial U'_{1j}}{\partial X_k}}.$$

Converting to momentum variables, we replace U'_{1i} by $m'_{1i}/\bar{\epsilon}_1$, which is valid to within our approximation (in the general case, we should write $u'_{1i}=m_{1i}/\epsilon_1-\overline{(m_{1i}/\epsilon_1)}$). We obtain

$$\bar{H}_{1ij}=4\nu_{t1}(W_{1ij}-D_{1ij}),$$

where

$$\nu_{t1}=a_1\omega^{-1},$$

$$W_{1ij}=-\frac{1}{4}\left[-\overline{\frac{\partial}{\partial X_k}\left(\frac{m'_{1k}}{\bar{\epsilon}_1}\right)\left(\frac{\partial}{\partial X_j}\left(\frac{m'_{1i}}{\bar{\epsilon}_1}\right)+\frac{\partial}{\partial X_i}\left(\frac{m'_{1j}}{\bar{\epsilon}_1}\right)\right)}\right.$$
$$\left.+\frac{2}{3}\overline{\frac{\partial}{\partial X_k}\left(\frac{m'_{1k}}{\bar{\epsilon}_1}\right)\left(\frac{\partial}{\partial X_j}\left(\frac{m'_{1i}}{\bar{\epsilon}_1}\right)+\frac{\partial}{\partial X_i}\left(\frac{m'_{1j}}{\bar{\epsilon}_1}\right)\right.}+\overline{\left.\delta_{ij}\frac{\partial}{\partial X_k}\left(\frac{m'_{1k}}{\bar{\epsilon}_1}\right)\right)}\right] \qquad [39]$$

and

$$D_{1ij}=\frac{1}{4}\left[\overline{\left(\frac{\partial}{\partial X_k}\left(\frac{m'_{1i}}{\bar{\epsilon}_1}\right)+\frac{\partial}{\partial X_i}\left(\frac{m'_{1k}}{\bar{\epsilon}_1}\right)\right)\left(\frac{\partial}{\partial X_k}\left(\frac{m'_{1j}}{\bar{\epsilon}_1}\right)+\frac{\partial}{\partial X_j}\left(\frac{m'_{1k}}{\bar{\epsilon}_1}\right)\right)}\right.$$
$$\left.+\overline{\frac{\partial}{\partial X_k}\left(\frac{m'_{1i}}{\bar{\epsilon}_1}\right)\frac{\partial}{\partial X_k}\left(\frac{m'_{1j}}{\bar{\epsilon}_1}\right)}-\overline{\frac{\partial}{\partial X_i}\left(\frac{m'_{1k}}{\bar{\epsilon}_1}\right)\frac{\partial}{\partial X_j}\left(\frac{m'_{1k}}{\bar{\epsilon}_1}\right)}\right].$$

Contracting, we obtain

$$D_{1ll} = \overline{\frac{1}{4}\left(\frac{\partial}{\partial X_k}\left(\frac{m'_{1l}}{\bar{\epsilon}_1}\right) + \frac{\partial}{\partial X_l}\left(\frac{m'_{1k}}{\bar{\epsilon}_1}\right)\right)\left(\frac{\partial}{\partial X_k}\left(\frac{m'_{1l}}{\bar{\epsilon}_1}\right) + \frac{\partial}{\partial X_l}\left(\frac{m'_{1k}}{\bar{\epsilon}_1}\right)\right)},$$

which is the positive energy-decay-rate tensor contraction, and

$$-W_{1ll} = -\frac{1}{3}\overline{\left[\frac{\partial}{\partial X_l}\left(\frac{m'_{1l}}{\bar{\epsilon}_1}\right)\right]^2},$$

which is a negative source term tensor contraction. The source term W_{1ll} produces irreversible gain of turbulence energy when the dilatation of the fluid is non-zero, which can occur in the two-field case despite the incompressibility of the fluid. Whereas W_{1ij} is of the lowest order in our model, W_{1ll} is of higher order. This means that there is not a source of energy in the case of the lowest order model, but only a redistribution of energy among the components of F_{1ij} and F_{2ij}.

3. CLOSURE MODELING

From now on, throughout this paper, we state that weak turbulence means circumstances in which the coupling between the phases is important (analogous to weak one-field turbulence in which closures depend on the viscosity ν), whereas intense turbulence means that closures are independent of C (just as the one-field closures become independent of ν).

We consider the problem of closure for [18]–[20], where we have added the dissipation terms. Throughout these derivations, we use techniques that have proven effective for the study of one-field turbulence. However, in contrast to the single-fluid case, we know that it is appropriate, for appreciable particle loading, to have closure based on the coupling parameter C, rather than on the viscosity of the fluid. Just as the one-fluid closures lose their dependence on molecular viscosity in the limit of high intensity turbulence, the two-field closures also lose their dependence on C. When the particles are sparsely dispersed, we have to add viscosity to [18]–[20] in order to get the right closure equations of turbulence in this limit.

The turbulent correlations we have to model are:

(1) correlations of pressure fluctuations with those of the momentum derivatives (e.g. $\overline{P'\partial m'_{1i}/\partial X_j}$);
(2) energy decay rates D_{1ij} and D_{2ij} and tensors W_{1ij} and W_{2ij};
(3) multiple correlations of various components of velocity fluctuations (e.g. $\overline{m'_{1i}m'_{1j}m'_{1k}}$, $\overline{m'_{2j}\partial/\partial X_k(m'_{1i}m'_{1k}/\bar{\epsilon}_1)}$).
(4) correlations of pressure fluctuations with those of the velocity $\overline{P'm'_{1i}}$.

3.1. *Closure for* $\overline{P'\left(\frac{\partial m'_{1j}}{\partial X_i} + \frac{\partial m'_{1i}}{\partial X_j}\right)}$

We have already described a closure for

$$\overline{P'\left(\frac{\partial m'_{1j}}{\partial X_i} + \frac{\partial m'_{1i}}{\partial X_j}\right)}$$

in [26] and [27]. Dimensional arguments suggest that this quantity may be proportional to the square root of the intensity of the turbulence. We propose here:

$$\overline{P'\left(\frac{\partial m'_{1j}}{\partial X_i} + \frac{\partial m'_{1i}}{\partial X_j}\right)} = k_1\rho_1\frac{\omega}{s}(\beta_1 F_{1kk} + \beta_2 F_{2kk})^{1/2}\left(\frac{\delta_{ij}}{3}F_{1kk} - F_{1ij}\right), \quad [40]$$

where s is the turbulence length scale, and k_1, β_1 and β_2 are coefficients of order unity. The closure for the corresponding term for the second fluid is obtained by exchanging the indices 1 and 2.

As in the one-field case, it is possible to describe a very simple closure for the energy-decay-rate tensor. We have noticed that W_{1kk} must vanish within the limits of our lowest order model. The reader is assured that the redistribution of energy among the components of F_{1ij} and F_{2ij} is taken into account in the closure [40].

3.2. Closure for D_{1ij} and D_{2ij}

For the decay terms D_{1ij} and D_{2ij}, we propose here a simple closure, where D_{1ij} and D_{2ij} are proportional to the Reynolds tensors F_{1ij} and F_{2ij}, respectively:

$$D_{1ij} = -\frac{1}{v_{t1}s}\psi\left(\frac{s}{s_0}\right)(\beta_{11}F_{1kk} + \beta_{12}F_{2kk})^{1/2}F_{1ij}, \quad \text{for the first field;} \qquad [41]$$

and

$$D_{2ij} = -\frac{1}{v_{t2}s}\psi\left(\frac{s}{s_0}\right)(\beta_{21}F_{1kk} + \beta_{22}F_{2kk})^{1/2}F_{2ij}, \quad \text{for the second field;}$$

where the βs are unspecified numbers of order unity, s_0 is some reference length scale, and ψ is a dimensionless function of (s/s_0). For the dissipation term in [20], we propose the following term:

$$-\frac{1}{v_t s}\psi\left(\frac{s}{s_0}\right)\left(\frac{F_{1kk}}{\bar{\epsilon}_1} + \frac{F_{2kk}}{\bar{\epsilon}_2} - \frac{2U_{kk}}{\bar{\epsilon}_1\bar{\epsilon}_2}\right)^{1/2} T_{ij},$$

which satisfiies the requirement that this term must vanish when the two fluids are identical. In the above expression we chose $v_t = s\sqrt{2q_{12}}$.

3.3. Closure for the multiple correlations

Define the quantities

$$A^1_{ijl} = \overline{m'_{1i}m'_{1j}m'_{1l}},$$

$$A^2_{ijl} = \overline{m'_{2i}m'_{2j}m'_{2l}},$$

$$B^1_{ijl} = \overline{m'_{1i}m'_{1j}m'_{2l}}$$

and

$$B^2_{ijl} = \overline{m'_{2i}m'_{2j}m'_{1l}}.$$

One can derive equations for A^1_{ijl}, A^2_{ijl}, B^1_{ijl} and B^2_{ijl} in a manner similar to that for the tensors F_{1ij} and F_{2ij} (see Besnard & Harlow 1985). For steady-state turbulence and homogeneous mean flow, the equation for A^1_{ijl} (and, similarly, for A^2_{ijl}) reduces, to the lowest order, to

$$\frac{1}{\rho_1}\left(\overline{m'_{1i}m'_{1l}\frac{\partial P'}{\partial X_j} + m'_{1j}m'_{1l}\frac{\partial P'}{\partial X_i} + m'_{1i}m'_{1j}\frac{\partial P'}{\partial X_l}}\right) - \frac{\partial}{\partial X_k}(F_{1ij}F_{1kl} + F_{1ik}F_{1jl} + F_{1jk}F_{1il}) + a_1(s)A^1_{ijl}$$
$$= -F_{1kl}\frac{\partial F_{1ij}}{\partial X_k} - F_{1jl}\frac{\partial F_{1ik}}{\partial X_k} - F_{1il}\frac{\partial F_{1jk}}{\partial X_k} + \frac{C}{\bar{\epsilon}_1\rho_1}[\bar{\epsilon}_1(B^1_{ilj} + B^1_{jil} + B^1_{lji}) - 3\bar{\epsilon}_2 A^1_{ijl}]. \qquad [42]$$

For the case of high intensity turbulence, which is likely to be independent of the coupling (as well as the viscosity), we neglect the coupling term in [42].

As a first step, we keep in [42] only those terms which we recognize to be described as self-diffusion of the turbulence in the equations for F_{1ij} and F_{2ij}, so that

$$A^1_{ijl} = -a_1(s)^{-1}\left(F_{1kl}\frac{\partial F_{1ij}}{\partial X_k} + F_{ijl}\frac{\partial F_{1ik}}{\partial X_k} + F_{1il}\frac{\partial F_{1jk}}{\partial X_k}\right),$$

and, in similar fashion,

$$A^2_{ijl} = -a_2(s)^{-1}\left(F_{2kl}\frac{\partial F_{2ij}}{\partial X_k} + F_{2jl}\frac{\partial F_{2ik}}{\partial X_k} + F_{2il}\frac{\partial F_{2jk}}{\partial X_k}\right). \qquad [43]$$

In the weak turbulence case, we do not neglect the coupling terms in [42] and have to solve a more complicated algebraic system in the variable A^1_{ijl}, A^2_{ijl}, B^1_{ijl} and B^2_{ijl}, which is presented in Besnard & Harlow (1985).

Expressions [43] are symmetric in i, j, l, as they should be (see Lumley 1978).

In order to get closure for the correlations $\overline{m'_{2j}\,\partial/\partial X_k(m'_{1k}m'_{1i}/\bar{\epsilon}_1)}$ and $\overline{m'_{1j}\,\partial/\partial X_k(m'_{2i}m'_{2k}/\bar{\epsilon}_2)}$, we multiply the momentum equations for m_{1i} and m_{1j} by m'_{1j} and m'_{1i}, respectively, and then add the resulting equations and take the derivative of the sum with respect to X_k. We then multiply the result by m'_{1k}. We also multiply the equation for m_{1k} by $\partial/\partial X_m(m'_{1i}m'_{ij})$ and add the result to the previous equation and take the ensemble average of that sum. The momentum equations are used to eliminate all the derivatives except $\overline{\partial/\partial t[m'_{2k}\,\partial/\partial X_m(m'_{1i}m'_{1j})]}$. We then make our usual assumptions of steady-state, homogeneous turbulence, and neglect the coupling terms and quadrupole correlations. As before, we ignore the pressure correlations for the sake of simplicity. Using the fact that the correlation $\overline{m'_{1i}(\partial m'_{2k}/\partial X_k)}$ is negligible within the limits of our approximation, we propose the closure

$$\overline{m'_{2j}\frac{\partial}{\partial X_k}\left(\frac{m'_{1i}m'_{1k}}{\bar{\epsilon}_1}\right)} = -\frac{s}{V}\left[\frac{\partial}{\partial X_k}\left(F_{2kl}\frac{\partial}{\partial X_l}\left(\frac{F_{1ij}\bar{\epsilon}_2}{\bar{\epsilon}_1}\right)\right)+\frac{\partial}{\partial X_k}\left(F_{1il}\frac{\partial}{\partial X_l}\left(\frac{T_{kj}}{\bar{\epsilon}_1}\right)\right)+\frac{\partial}{\partial X}\left(F_{1jl}\frac{\partial}{\partial X_l}\left(\frac{T_{ik}}{\bar{\epsilon}_1}\right)\right)\right], \quad [44]$$

with $V = (\beta_1 F_{1kk} + \beta_2 F_{2kk})^{1/2}$.

The expression for the correlation $\overline{m'_{1j}\,\partial/\partial X_k(m'_{2i}m'_{2k}/\bar{\epsilon}_2)}$ is obtained by exchanging indices 1 and 2.

3.4. Pressure velocity correlations $\overline{P'm'_{1i}}$ and $\overline{P'm'_{2i}}$

If we start from the equation for m_{1j}, and take its divergence with respect to x, neglecting the higher order term in $\partial^2\epsilon'_i/\partial t^2$, then apply Green's theorem on the fluctuating pressure and multiply the result by m'_{1i} and finally take an ensemble average, we obtain

$$\overline{m'_{1i}P'(\mathbf{X})} = +\frac{\rho_1}{4\pi}\int \overline{m'_{1i}(\mathbf{X})\frac{\partial}{\partial X_j}\left[\frac{1}{\epsilon_1}\frac{\partial}{\partial X_k}\left(\frac{m_{1j}m_{1k}}{\epsilon_1}\right)-\frac{C}{\rho_1\epsilon_1}(\epsilon_1 m_{2j}-\epsilon_2 m_{1j})\right](\mathbf{X}')}\frac{dX'}{r}, \quad [45]$$

where $r = \|\mathbf{X} - \mathbf{X}'\|$, and the integration is performed over the whole space. However, because the fluctuating momenta are not correlated over distances exceeding the largest scale length of the turbulence, we can restrict the integration to a finite volume centered in $\mathbf{X}$, and of radius s.

For the lowest order part of the integral (in terms of ϵ'_1), we propose:

$$\overline{m'_{1i}P'} = -\left(C_{P1}sF^{1/2}_{1kk}\frac{\partial F_{1il}}{\partial X_l} + C_{P2}s^2\frac{\partial}{\partial X_l}\left(\frac{\overline{m}_{1l}}{\bar{\epsilon}_1}\right)\frac{\partial F_{1ik}}{\partial X_k}\right), \quad [46]$$

using the assumption that the mean variables can be considered as slowly varying over the distance s.

Thus, the conservative pressure correlation

$$\left[\frac{\partial}{\partial X_j}\overline{(m'_{1i}P')} + \frac{\partial}{\partial X_i}\overline{(m'_{1j}P')}\right]$$

can be decomposed into two parts, self-diffusion and diffusion due to the mean shear stress.

As a summary, we give in appendix A the model equations for F_{1ij}, F_{2ij} and T_{ij}, which include all the previous closures.

4. INSTABILITIES OF THE TWO-FIELD FLOW

The goal of this section is to relate the instabilities of a two-field flow to the developed turbulence. We analyze the problem of coherent instability of the fluid equations for a two-field flow in the absence of the viscous energy dissipation.

When a steady-state solution exists, two possible interpretations can be given. One of these considers that the flow field consists entirely of a time-varying mean, for which there is conceivably a steady state. The steady state, however, is unstable in the absence of viscous energy dissipation.

The other interpretation considers the ensemble average of the fluctuating flow to be the mean-flow part, with the rest described as turbulence. Any perturbation to the mean flow is transformed into fluctuational energy. We show, using this fact, that the turbulent stress tensor can stabilize both the linear and non-linear perturbations to the flow. Here F_{1ij} plays the same role as the collisional pressure in Needham & Nerkin (1983). With a linear analysis of the equations in the absence of identifiable turbulence, we derive the rate at which mean-flow energy is transformed into fluctuational energy. The next step is to identify the fluctuational part of the flow as turbulence and to use the energy transformation rate as a source to the turbulence energy. The third step is then to use the energy transformation rate as a source to the turbulence energy.

At the end of this section, we show that this source term to the turbulence energy can be exhibited directly from the Reynolds tensor transport equation through a careful examination of the different pressure correlations.

To study the coherent instability of the non-turbulent two-field flow, we linearize [7]–[11], to which we have added the gravity terms. The zeroth-order solution is chosen to be the steady state for a fluidized dust bed, in which the pressure gradient balances the overall hydrostatic gravitational force. From the resulting dispersion relation we deduce the growth rate ω^* of a perturbation of wavenumber k. In the weak coupling case, ω^* is proportional to the zeroth-order mean velocity difference, $|u_1^0 - u_2^0|$. In the strong coupling case, ω^* is proportional to $(u_1^0 - u_2^0)^2/C$. The fluctuational energy E_f contained in any disturbance grows according to the equation $\mathrm{d}E_f/\mathrm{d}t = 2\omega^* E_f$.

Consider now equations for turbulent two-field flow. Equations [12]–[14] and [21]–[22] contain additional terms, when compared to [7]–[11] due to the presence of turbulence. We include as a source in the turbulence energy equations this same amount of fluctuational energy, which, in turn, will be balanced by the decay terms.

In this interpretation, the investigation of stability again introduces perturbations into the mass and momentum equations for the mean flow. The turbulence energy equations, however, already represent the effect of the fluctuational part of the dynamics, and, accordingly, are not perturbed in this analysis.

It was found by Besnard & Harlow (1985) that, for given disturbance of wavenumber k, there exists a certain level of turbulence above which this perturbation is stabilized.

Define the quantities

$$H = \rho_1 \frac{q_1}{\epsilon_1^{02}} + \rho_2 \frac{q_2}{\epsilon_2^{02}},$$

$$A_r = \rho_1 \epsilon_2^0 + \rho_2 \epsilon_1^0,$$

$$B_r = C + \left(\rho_1 \nu_{t1} \frac{\epsilon_2^0}{\epsilon_1^0} + \rho_2 \nu_{t2} \frac{\epsilon_1^0}{\epsilon_2^0}\right) k^2,$$

$$B_i = 2k(\epsilon_1^0 \rho_2 u_2^0 + \epsilon_2^0 \rho_1 u_1^0)$$

and

$$C_i = k\left[C(\epsilon_1^0 u_1^0 + \epsilon_2^0 u_2^0) + k^2\left(\rho_1 \nu_{t1} \frac{\epsilon_2^0}{\epsilon_1^0} u_1^0 + \rho_2 \nu_{t2} \frac{\epsilon_1^0}{\epsilon_2^0} u_2^0\right)\right].$$

The level of turbulence which stabilizes the disturbance is obtained through the condition

$$H = \epsilon_2^0 u_1^{02} \rho_1 + \epsilon_1^0 \rho_2 u_2^{02} + \frac{C_i}{k^2 B_r}(A_r C_i - B_r B_i), \qquad [47]$$

since H depends on the turbulence intensities q_1 and q_2.

The intensity of turbulence which is retained must be a steady-state solution of the turbulence [A.1]–[A.3] (see the appendix) and balance exactly the loss of mean flow. For the sake of simplicity, we only retain a few terms in these equations. We have

$$\left.\begin{aligned} &2\frac{C}{\rho_1}(q_{12}-\epsilon_2 q_1)-\lambda\frac{q_1^{1/2}}{s}q_1+2\omega^* q_1=0,\\ &2\frac{C}{\rho_2}(q_{12}-\epsilon_1 q_2)-\lambda\frac{q_2^{1/2}}{s}q_2+2\omega^* q_2=0,\\ &C\frac{\epsilon_2}{\rho_1}(\epsilon_1 q_2-q_{12})+C\frac{\epsilon_1}{\rho_2}(\epsilon_2 q_1-q_{12})=0. \end{aligned}\right\} \quad [48]$$

The terms with λ are simplified versions of [41]. Consistency between [47] and [48] determines a smallest value for the parameter λ. It is shown in Besnard & Harlow (1985) that the system [48] always has a positive solution (q_1, q_2), as expected.

This linear analysis supports the direct derivation of a source term proportional to the mean velocity difference between the two fields. In this case of transition from instabilities towards turbulence, the developing turbulence is very small at the early stages of the process and our assumption of negligible correlations in ϵ_1' is no longer valid. Then we consider the term in [45] which is proportional to $C(\epsilon_1 m_{2j}-\epsilon_2 m_{1j})$.

Equation [46], therefore, presents an additional term which we model as

$$\overline{m_{1i}'P_C'}=C_{P3}C\frac{s^2}{\bar{\epsilon}_1^2}\overline{m_{1i}'\frac{\partial\epsilon_1'}{\partial X_j}}(\bar{\epsilon}_1\bar{m}_{2j}-\bar{\epsilon}_2\bar{m}_{1j}). \quad [49]$$

The correlation $\overline{m_{1i}'\partial\epsilon_1'/\partial X_j}$ can be modeled as in Lumley (1978). For the sake of simplicity, we take a gradient-type approximation for it and [49] becomes

$$\overline{m_{1i}'P_C'}=-C_{P4}Cs^2F_{1ik}\frac{\partial^2\bar{\epsilon}_1}{\partial X_k\,\partial X_j}\left(\frac{\bar{m}_{2j}}{\bar{\epsilon}_2}-\frac{\bar{m}_{1j}}{\bar{\epsilon}_1}\right). \quad [50]$$

As a consequence, the conservative pressure correlation

$$\frac{\partial}{\partial X_j}\overline{(m_{1i}'P')}+\frac{\partial}{\partial X_i}\overline{(m_{1j}'P')}$$

includes additional terms, among them source terms proportional to the mean velocity difference between the two fields, as expected.

The full system of tensors [A.1]–[A.3] in the appendix will be solvable analytically only for very simple circumstances, and even the use of numerical techniques will often prove difficult for complicated problems. From this set of equations, it is possible to derive a simplified model, like the "k–ϵ" model for single-fluid turbulence developed by Launder *et al.* (1975). The complete derivation of such a model is not discussed in this paper, since it is beyond the scope of this work. However, even at this stage of complexity, it is possible to give an interpretation of the model for some limiting cases.

5. LIMITING CASES

The goals of this section are to interpret [A.1]–[A.3] (see the appendix) in terms of their consequences for several simple problems. These include the diffusion and wave-like transport of the volume fraction, the coupling decay of homogeneous turbulence and the decay of turbulence in a gas as a result of inertial loading by imbedded particles. Simple analysis of these problems can neglect the effects of the triple correlation and the pressure correlations and the source and dissipation terms in [A.1]–[A.3].

Considering [A.1]–[A.3], together with the equation for the mean volume fractions, and mean momenta, and dropping the overbars for convenience, we have the following set of equations:

$$\frac{\partial\epsilon_1}{\partial t}+\frac{\partial m_{1k}}{\partial X_k}=0, \quad [51]$$

$$\frac{\partial \epsilon_2}{\partial t} + \frac{\partial m_{2k}}{\partial X_k} = 0, \quad [52]$$

$$\frac{\partial m_{1i}}{\partial t} + \frac{\partial}{\partial X_k}\left(\frac{m_{1i}m_{1k}}{\epsilon_1}\right) + \frac{\partial F_{1ik}}{\partial X_k} = \frac{C}{\rho_1}(\epsilon_1 m_{2i} - \epsilon_2 m_{1i}) - \frac{\epsilon_1}{\rho_1}\frac{\partial P}{\partial X_i}, \quad [53]$$

$$\frac{\partial m_{2i}}{\partial t} + \frac{\partial}{\partial X_k}\left(\frac{m_{2i}m_{2k}}{\epsilon_2}\right) + \frac{\partial F_{2ik}}{\partial X_k} = \frac{C}{\rho_2}(\epsilon_2 m_{1i} - \epsilon_1 m_{2i}) - \frac{\epsilon_2}{\rho_2}\frac{\partial P}{\partial X_i}, \quad [54]$$

$$\frac{\partial \epsilon_1 F_{1ij}}{\partial t} + \frac{\partial m_{1k}F_{1ij}}{\partial X_k} + \epsilon_1\left[F_{1ij}\frac{\partial}{\partial X_k}\left(\frac{m_{1k}}{\epsilon_1}\right) + F_{1kj}\frac{\partial}{\partial X_k}\left(\frac{m_{1i}}{\epsilon_1}\right) + F_{1ik}\frac{\partial}{\partial X_k}\left(\frac{m_{1j}}{\epsilon_1}\right)\right]$$

$$= 2C\frac{\epsilon_1}{\rho_1}(U_{ij} - \epsilon_2 F_{1ij}), \quad [55]$$

$$\frac{\partial \epsilon_2 F_{2ij}}{\partial t} + \frac{\partial m_{2k}F_{2ij}}{\partial X_k} + \epsilon_2\left[F_{2ij}\frac{\partial}{\partial X_k}\left(\frac{m_{2k}}{\epsilon_2}\right) + F_{2kj}\frac{\partial}{\partial X_k}\left(\frac{m_{2i}}{\epsilon_2}\right) + F_{2ik}\frac{\partial}{\partial X_k}\left(\frac{m_{2i}}{\epsilon_2}\right)\right]$$

$$= 2C\frac{\epsilon_2}{\rho_2}\left(U_{ij} - \epsilon_1 F_{2ij}\right) \quad [56]$$

and

$$\frac{\partial T_{ij}}{\partial t} + \tilde{U}_k\frac{\partial T_{ij}}{\partial X_k} + T_{ij}\frac{\partial}{\partial X_k}\left(\frac{m_{1k}}{\epsilon_1} + \frac{m_{2k}}{\epsilon_2}\right) + T_{ij}\frac{\partial}{\partial X_k}\left(\frac{m_{1k}}{\epsilon_1} + \frac{m_{2k}}{\epsilon_2}\right) + T_{kj}\frac{\partial}{\partial X_k}\left(\frac{m_{1i}}{\epsilon_1}\right)$$

$$+ T_{ik}\frac{\partial}{\partial X_k}\left(\frac{m_{2j}}{\epsilon_2}\right) = C\frac{\epsilon_2}{\rho_1}(\epsilon_1 F_{2ij} - T_{ij}) + C\frac{\epsilon_1}{\rho_2}(\epsilon_2 F_{1ij} - T_{ij}). \quad [57]$$

5.1. The strong coupling limit

To non-dimensionalize this system, we scale according to the following dimensionful quantities: time T, distance X, volume flux M (such that $MT = X$), pressure P, density R, pressure RM^2 and coupling T/R. The system [51]–[57] is formally invariant, and, for convenience, we consider these equations for the strong coupling case $C \gg 1$. This means that the coupling between the two fluids is the dominant process. In this limit we ordinarily expect the two mean-flow velocities to approach each other, $m_{1i}/\epsilon_2 \leftrightarrow m_{2i}/\epsilon_1$. Our purpose here, however, is to show that the presence of turbulence can alter this expectation, in particular to produce a balance between the gradient of turbulence energy and the persistence of a non-zero mean-flow interpenetration. This paradox is not only consistent with the equations, but it is also intuitively plausible when we recognize that microscopic tight coupling does not preclude the presence of turbulence, which can mix the fields in the presence of a volume fraction gradient. This circumstance accordingly represents the diffusion limit. We thus explore the possibility of the following expansion in the small parameter C^{-1}:

$$F_{1ij} = C\sum_n C^{-n}F_{1ij}^{(n)}; \quad \epsilon_1 = \sum C^{-n}\epsilon_1^{(n)}; \quad m_{1i} = \sum C^{-n}m_{1i}^{(n)};$$

$$F_{2ij} = C\sum_n C^{-n}F_{2ij}^{(n)}; \quad \epsilon_2 = \sum C^{-n}\epsilon_2^{(n)}; \quad m_{2i} = \sum c^{-n}m_{2i}^{(n)}; \quad P = \sum C^{-n}p^{(n)};$$

$$T_{ij} = C\sum C^{-n}T_{ij}^{(n)}; \quad U_{ij} = C\sum C^{-n}U_{ij}^{(n)}.$$

To the lowest order in (C^{-1}), [51]–[57] become

$$\frac{\partial \epsilon_1^{(0)}}{\partial t} + \frac{\partial m_{1k}^{(0)}}{\partial X_k} = 0, \quad [58]$$

$$\frac{\partial \epsilon_2^{(0)}}{\partial t} + \frac{\partial m_{2k}^{(0)}}{\partial X_k} = 0, \quad [59]$$

$$\frac{\partial F^{(0)}_{1ik}}{\partial X_k} = \frac{1}{\rho_1}(\epsilon_1^{(0)} m_{2i}^{(0)} - \epsilon_2^{(0)} m_{1i}^{(0)}), \qquad [60]$$

$$\frac{\partial F^{(0)}_{2ik}}{\partial X_k} = \frac{1}{\rho_2}(\epsilon_2^{(0)} m_{1i}^{(0)} - \epsilon_1^{(0)} m_{2i}^{(0)}), \qquad [61]$$

$$U_{ij}^{(0)} = \epsilon_2^{(0)} F_{1ij}^{(0)}, \qquad [62]$$

$$U_{ij}^{(0)} = \epsilon_1^{(0)} F_{2ij}^{(0)} \qquad [63]$$

and

$$T_{ij}^{(0)} = T_{ji}^{(0)} = U_{ij}^{(0)}. \qquad [64]$$

Equations [58] and [59] give

$$\frac{\partial}{\partial X_k}(m_{1k}^{(0)} + m_{2k}^{(0)}) = 0,$$

so that we obtain a momentum conservation equation

$$m_{1k}^{(0)} + m_{2k}^{(0)} = (\text{curl}\, \boldsymbol{\psi})_k,$$

in which $\boldsymbol{\psi}$ is an arbitrary vector independent of time.

Equations [60] and [61] become

$$\frac{\partial}{\partial X_k}(\rho_1 F_{1ik}^{(0)}) = -m_{1i}^{(0)} + \epsilon_1^{(0)}(\text{curl}\, \boldsymbol{\psi})_i$$

and

$$\frac{\partial}{\partial X_k}(\rho_2 F_{2ik}^{(0)}) = -m_{2i}^{(0)} + \epsilon_2^{(0)}(\text{curl}\, \boldsymbol{\psi})_i.$$

By addition, we obtain

$$\frac{\partial}{\partial X_k}(\rho_1 F_{1ik}^{(0)} + \rho_2 F_{2ik}^{(0)}) = 0,$$

so that

$$\rho_1 F_{1ik}^{(0)} + \rho_2 F_{2ik}^{(0)} = \boldsymbol{\Phi}_{ik}, \qquad [65]$$

in which $\boldsymbol{\Phi}_{ik}$ is an arbitrary tensor independent of time with vanishing divergence. This equation describes the conservation of turbulence energy; it also represents a consistency condition between the initial turbulence energy levels of the two fields in the limit of strong microscopic coupling.

Using [62] and [63] and the condition $\epsilon_1^{(0)} + \epsilon_2^{(0)} = 1$, [60] becomes

$$\frac{\partial F_{1ij}}{\partial X_j} = \frac{1}{\epsilon_2^{(0)}}\frac{\partial U_{ij}^{(0)}}{\partial X_j} + \frac{U_{ij}}{(\epsilon_2^{(0)})^2}\frac{\partial \epsilon_1^{(0)}}{\partial X_j},$$

so that

$$m_{1i}^{(0)} = -\rho_1\left(\frac{1}{\epsilon_2^{(0)}}\frac{\partial U_{ij}^{(0)}}{\partial x_j} + \frac{U_{ij}^{(0)}}{(\epsilon_2^{(0)})^2}\frac{\partial \epsilon_1^{(0)}}{\partial X_j}\right) + \epsilon_1^{(0)} B_i, \qquad [66]$$

where $B_i = (\text{curl}\, \boldsymbol{\psi})_i$ represents the mean translational velocity in any homogeneous region.

With [62], [63] and [65], we can solve for $U_{ij}^{(0)}$ and insert the result in [66]. This, in turn can be combined with [58] to give

$$\frac{\partial \epsilon_1^{(0)}}{\partial t} = \rho_1\rho_2 \frac{\partial}{\partial X_k}\left(\frac{\boldsymbol{\Phi}_{kl}}{\rho_1\epsilon_1^{(0)} + \rho_2\epsilon_2^{(0)}}\frac{\partial \epsilon_1^{(0)}}{\partial X_l}\right) - \frac{\partial}{\partial X_k}(\epsilon_1^{(0)} B_k). \qquad [67]$$

This equation for $\epsilon_1^{(0)}$ includes both convective and diffusive parts. In the special case where $B = 0$, this formulation shows the result that other authors have derived in a very different manner, namely, that the diffusion of the fluids can be driven by the turbulent kinetic energy (e.g. Lumley 1975, 1978).

This result also can be used to support the gradient approximations that are proposed for closure of the turbulence equations.

5.2. The weak coupling limit

The above behavior only occurs if the dominant phenomenon is the coupling between the two fields, as may occur, for example, with very small and/or lightweight particles in a fluid. For other circumstances of multiphase flow, the coupling can be considered weak enough that $C \ll 1$. Neglecting terms including C, we rewrite [51]–[58] for a one-dimensional geometry:

$$\epsilon_1 + \epsilon_2 = 1,$$

$$\frac{\partial \epsilon_1}{\partial t} + \frac{\partial m_1}{\partial X} = 0,$$

$$\frac{\partial \varepsilon_2}{\partial t} + \frac{\partial m_2}{\partial X} = 0,$$

$$\frac{\partial m_1}{\partial t} + \frac{\partial}{\partial X}\left(\frac{m_1^2}{\epsilon_1}\right) + \frac{\partial F_1}{\partial X} + \frac{\epsilon_1}{\rho_1}\frac{\partial P}{\partial X} = 0,$$

$$\frac{\partial m_2}{\partial t} + \frac{\partial}{\partial X}\left(\frac{m_2^2}{\epsilon_2}\right) + \frac{\partial F_2}{\partial X} + \frac{\epsilon_2}{\rho_2}\frac{\partial P}{\partial X} = 0,$$

$$\frac{\partial \epsilon_1 F_1}{\partial t} + \frac{\partial m_1 F_1}{\partial x} + 3\epsilon_1 F_1 \frac{\partial}{\partial X}\left(\frac{m_1}{\epsilon_1}\right) = 0$$

and

$$\frac{\partial \epsilon_2 F_2}{\partial t} + \frac{\partial m_2 F_2}{\partial X} + 3\epsilon_2 F_2 \frac{\partial}{\partial X}\left(\frac{m_2}{\epsilon_2}\right) = 0.$$

Thus the coupling between the two fields is accomplished only through the pressure term and the condition $\epsilon_1 + \epsilon_2 = 1$. Studying the problem defined in figure 1, it can be shown, in the limiting case $\epsilon_1/\rho_1 \ll \epsilon_2/\rho_2$, that the velocity of the turbulent front is then $m_1/\epsilon_1 \pm (3F_1/\epsilon_1)^{1/2}$.

The characteristic parameters of the solution are

$$\left.\begin{aligned}
\epsilon_{1m} &= \tfrac{1}{2}(\epsilon_{1R} + \epsilon_{1L}),\\
F_{1m} &= \tfrac{\alpha}{8}(\epsilon_{1R} + \epsilon_{1L})^3,\\
m_{1m} &= 3\alpha(\epsilon_{1R} - \epsilon_{1L}),\\
\xi_1 &= \frac{m_{1m}}{\epsilon_{1m}} + \left(3\frac{F_{1m}}{\epsilon_{1m}}\right)^{1/2},\\
\xi_2 &= \frac{m_{1m}}{\epsilon_{1m}} - \left(3\frac{F_{1m}}{\epsilon_{1m}}\right)^{1/2},\\
\xi_R &= \frac{m_{1m}}{\epsilon_{1m}} + \left(3\frac{F_{1R}}{\epsilon_{1R}}\right)^{1/2},\\
\xi_L &= \frac{m_{1m}}{\epsilon_{1m}} - \left(3\frac{F_{1L}}{\epsilon_{1L}}\right)^{1/2}.
\end{aligned}\right\} \qquad [68]$$

For initial conditions consisting of a step function in ϵ_1 centered at $X = 0$, with m being zero for all positive values of X, the subsequent behavior of the fluid can be described as the propagation of a wave in each direction. The front of each wave is of finite extension $(\xi_R - \xi_2)t$ and $(\xi_1 - \xi_L)t$.

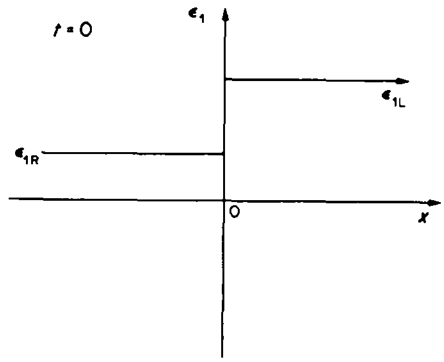

Figure 1. Initial condition for the volume fraction ϵ_1 in the weak coupling limit; ϵ vs the space coordinate x.

Within the two transition regions, the solution is described by the equations:

$$\left.\begin{aligned} F_1 &= -\tfrac{1}{8}(3\epsilon_L)^{-1/6}(\xi - \beta_L)^3, \\ \epsilon_1 &= -\tfrac{1}{2}(3\epsilon_L)^{-1/2}(\xi - \beta_L), \\ m_1 &= \tfrac{1}{2}(3\epsilon_L)^{-1/2}(\xi - \beta_L)\left(\beta_L + \frac{\xi - \beta_L}{2}\right), \end{aligned}\right\} \quad [69]$$

in which $\epsilon_L = F_{1L}\epsilon_{1L}^{-3}$ and $\beta_L = m_{1L}/\epsilon_{1L} + (3\alpha)^{1/2}\epsilon_{1L}$, in the left transition region; and

$$\left.\begin{aligned} F_1 &= -\tfrac{1}{8}(3\epsilon_R)^{1/2}(\xi - \beta_R)^3, \\ \epsilon_1 &= \tfrac{1}{2}(3\epsilon_R)^{-1/2}(\xi - \beta_R), \\ m_1 &= \tfrac{1}{2}(3\epsilon_R)^{-1/2}(\xi - \beta_R)\left(\beta_R - \frac{\xi - \beta_R}{2}\right), \end{aligned}\right\} \quad [70]$$

in which $\epsilon_R = F_{1R}\epsilon_{1R}^{-3}$ and $\beta_R = m_{1R}/\epsilon_{1R} - (3\alpha)^{1/2}\epsilon_{1R}$, in the right transition region (figure 2).

In the more general case for which ρ_1 is neither small nor large compared with ρ_2, the solutions are more complicated but nevertheless also exhibit this wave-like behavior in the weak coupling limit under consideration.

The results of this and the preceding section resemble similar results from the investigation of two interpenetrating gases initially separated by a membrane. The random (Maxwell–Boltzmann) velocity fluctuations correspond to the turbulence in our two-fluid model. If the mean-free-path of the molecules is large, then there is negligible coupling between the two interpenetrating gases, and each propagates as though freely expanding into a vacuum, with wave-like rarefactions going in all directions from the initial point of contact. If the mean-free-path is short, then there is strong

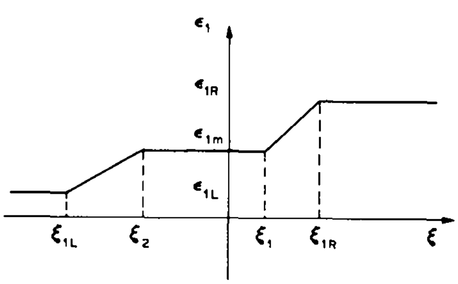

Figure 2. Self-similar solution for the volume fraction in the weak coupling limit; ϵ vs the self-similar variable $\xi = x/t$.

coupling and the interpenetration becomes a diffusive process. In both extremes there is a net mean velocity in each of the gases, just as we obtain in our two-fluid turbulence model based on m and ϵ. A model based on u and ϵ would not show these two distinct interpenetration velocities, and would exhibit only the diffusion limit, as discussed in section 1.

Available models usually use a diffusive-like evolution equation for ϵ, and therefore would not show these two distinct interpenetration velocities, as discussed in section 1.

5.3. *Decay of turbulence*

There are several important features of our turbulence model that can be exhibited by a consideration of the decay in homogeneous circumstances without source terms. In particular, we examine the late stages of decay, when the coupling effects are dominant, rather than the cascade decay of energy when the turbulence is more intense. We consider the system [55]–[57]. With homogeneity, the different components are decoupled. We obtain the simple system

$$\frac{\partial F_{1ij}}{\partial t} = \frac{2C}{\rho_1}(U_{ij} - \epsilon_2 F_{1ij}), \tag{71}$$

$$\frac{\partial F_{2ij}}{\partial t} = \frac{2C}{\rho_2}(U_{ij} - \epsilon_1 F_{2ij}) \tag{72}$$

and

$$\frac{\partial U_{ij}}{\partial t} = \frac{C\epsilon_2}{\rho_1}(\epsilon_1 F_{2ij} - U_{ij}) + \frac{C\epsilon_1}{\rho_2}(\epsilon_2 F_{1ij} - U_{ij}), \tag{73}$$

where ϵ_1 and ϵ_2 are constants.

With $U_{ij} = \overline{m'_{1i} m'_{2j}}$, $F_{1ij} = \overline{m'_{1i} m'_{1j}/\epsilon_1}$ and $F_{2ij} = \overline{m'_{2i} m'_{2j}/\epsilon_2}$ we can combine [71]–[73] to show that

$$\frac{\partial}{\partial t}(\rho_1 F_{1ij} + \rho_2 F_{2ij}) = 2C\epsilon_1\epsilon_2(2U_{ij} - F_{1ij}\epsilon_2 - F_{2ij}\epsilon_1) = 2C\epsilon_1\epsilon_2 \overline{\left(\frac{m'_{2j}}{\epsilon_2} - \frac{m'_{1j}}{\epsilon_1}\right)\left(\frac{m'_{2i}}{\epsilon_2} - \frac{m'_{1i}}{\epsilon_1}\right)}.$$

This shows that the contraction of the tensor $(2U_{ij} - \epsilon_1 F_{2ij} - \epsilon_2 F_{1ij})$ is $\geqslant 0$, so that turbulent kinetic energy per unit volume, $\rho_1 F_{1ii} + \rho_2 F_{2ii}$ monotonically decays in this circumstance. In addition, we can combine [71]–[73] to obtain

$$\frac{\partial}{\partial t}(2U_{ij} - \epsilon_1 F_{2ij} - \epsilon_2 F_{1ij}) = -2C\left(\frac{\epsilon_1}{\rho_2} + \frac{\epsilon_2}{\rho_1}\right)(2U_{ij} - \epsilon_1 F_{2ij} - \epsilon_2 F_{1ij}).$$

This shows that the components of this tensor also decay monotonically, each in proportion to itself, suggesting a tendency for any arbitrary distribution of turbulence between the two fields to approach a specific apportionment in which $2U_{ij} - \epsilon_1 F_{2ij} - \epsilon_2 F_{1ij} = 0$. However, if an initially specified apportionment is such that $U_{ij} = \epsilon_1 F_{2ij} = \epsilon_2 F_{1ij}$, then the turbulence does not decay as a result of coupling, and would then dissipate only as a result of the presently neglected viscosity.

6. CONCLUDING REMARKS

In this paper, we have developed the first stage of a two-field turbulence model, which rests on the rigorous derivation of the turbulence energy equations from the two-phase fluid equations and the provision of a closure of these equations. This paper presented the Reynolds averaged set of transport equations for incompressible two-phase flow. The volume fraction of the dispersed phase must be small enough that the assumption of no interaction between the particles is valid. However, this restriction does not imply that the dispersed phase must be sparsely dispersed. Much remains to be accomplished. Of interest is the validity of the low-order approximations that we have described in this paper.

We have derived in section 2 the decay term for our model by considering the cascade of energy from large-scale turbulence to small-scale turbulence.

Section 3 presents the closures for all of the correlations involved in the momentum equations and in the turbulent field equations. In many cases the closure relies strongly on the inclusion of decay, either the cascade decay or the coupling decay. These terms are especially important for triple correlations, for which these decay terms have not been rigorously derived. Also, the modeling of the pressure correlation terms are not yet fully understood. In any case, whatever models are proposed will have to be substantiated by comparisons with experiments.

Section 5 discusses several interpretative features of our model. For example, in the case of strong coupling we obtain the model of other investigators for the diffusion of the volume fraction. In contrast to the previous models, we also show a weak-coupling, wave-like limit similar to that of diffusing dispersed gases, and we give a precise estimate for the velocity of such waves. We have also examined the properties of turbulence decay as a result of coupling between fluids. In each case, the analysis has utilized numerous approximations, which will require further investigation.

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APPENDIX

$$
\begin{aligned}
&\frac{\partial F_{1ij}}{\partial t}+\frac{\overline{m_{1k}}}{\bar{\epsilon}_1}\frac{\partial F_{1ij}}{\partial X_k}+F_{1ij}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1k}}}{\bar{\epsilon}_1}\right)+F_{1ik}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1j}}}{\bar{\epsilon}_1}\right)+F_{1kj}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1i}}}{\bar{\epsilon}_1}\right)\\
&\quad=\frac{2C}{\rho_1}(U_{ij}-\bar{\epsilon}_2F_{1ij})+k_1\rho_1\frac{\omega}{s}(\beta_1F_{1kk}+\beta_2F_{2kk})^{1/2}\left(\frac{\delta_{ij}}{3}F_{1kk}-F_{1ij}\right)\\
&\qquad+\frac{\partial}{\partial X_k}\left[(a_1(s)\bar{\epsilon}_1^2)^{-1}\left(F_{1kl}\frac{\partial F_{1ij}}{\partial X_l}+F_{1jk}\frac{\partial F_{1il}}{\partial X_l}+F_{1ik}\frac{\partial F_{1jl}}{\partial X_l}\right)\right]\\
&\qquad+\frac{\partial}{\partial X_j}\left[C_{P1}F_{1kk}^{1/2}\frac{\partial F_{1il}}{\partial X_l}+C_{P2}s^2\frac{\partial}{\partial X_l}\left(\frac{\bar{m}_{1l}}{\bar{\epsilon}_1}\right)\frac{\partial F_{1ik}}{\partial X_k}\right.\\
&\qquad\left.+C_{P4}s^2F_{1ik}\frac{\partial^2\bar{\epsilon}_1}{\partial X_k\,\partial X_l}C\left(\frac{\bar{m}_{2l}}{\bar{\epsilon}_2}-\frac{\bar{m}_{1l}}{\bar{\epsilon}_1}\right)\right]
\end{aligned}
$$

$$+\frac{\partial}{\partial X_i}\left[C_{P1}sF_{1kk}^{1/2}\frac{\partial F_{1jl}}{\partial X_l}+C_{P_2}s^2\frac{\partial}{\partial X_l}\left(\frac{\bar{m}_{1l}}{\bar{\epsilon}_1}\right)\frac{\partial F_{1jk}}{\partial X_k}\right.$$
$$\left.+C_{P4}s^2F_{1jk}\frac{\partial^2\bar{\epsilon}_1}{\partial X_kX_l}C\left(\frac{\bar{m}_{2l}}{\bar{\epsilon}_2}-\frac{\bar{m}_{1l}}{\bar{\epsilon}_1}\right)\right]-4\frac{\psi}{s}(\beta_{11}F_{1kk}+\beta_{12}F_{2kk})^{1/2}F_{1ij}, \qquad [A.1]$$

$$\frac{\partial F_{2ij}}{\partial t}+\frac{\overline{m_{2k}}}{\bar{\epsilon}_2}\frac{\partial F_{2ij}}{\partial X_k}+F_{2ij}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{2k}}}{\bar{\epsilon}_2}\right)+F_{2ik}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{2j}}}{\bar{\epsilon}_2}\right)+F_{1kj}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{2i}}}{\bar{\epsilon}_2}\right)$$
$$=\frac{2C}{\rho_2}(U_{ij}-\bar{\epsilon}_1F_{2ij})+k_2\rho_2\frac{\omega}{s}(\beta_1F_{1kk}+\beta_2F_{2kk})^{1/2}\left(\frac{\delta_{ij}}{3}F_{2kk}-F_{2ij}\right)$$
$$+\frac{\partial}{\partial X_k}\left[(a_2(s)\bar{\epsilon}_1^2)^{-1}\left(F_{2kl}\frac{\partial F_{2ij}}{\partial X_l}+F_{2jk}\frac{F_{2il}}{\partial X_l}+F_{2ik}\frac{\partial F_{2jl}}{\partial X_l}\right)\right]$$
$$+\frac{\partial}{\partial X_j}\left[C_{P1}sF_{2kk}^{1/2}\frac{\partial F_{2il}}{\partial X_l}+C_{P2}s^2\frac{\partial}{\partial X_l}\left(\frac{\overline{m_{2l}}}{\bar{\epsilon}_2}\right)\frac{\partial F_{2ik}}{\partial X_k}\right.$$
$$\left.+C_{P4}s^2F_{2ik}\frac{\partial^2\bar{\epsilon}_2}{\partial X_k\partial X_l}C\left(\frac{\overline{m_{1l}}}{\bar{\epsilon}_1}-\frac{\overline{m_{2l}}}{\bar{\epsilon}_2}\right)\right]$$
$$+\frac{\partial}{\partial X_i}\left[C_{P1}sF_{2kk}^{1/2}\frac{\partial F_{2jl}}{\partial X_l}+C_{P2}s^2\frac{\partial}{\partial X_l}\left(\frac{\overline{m_{2l}}}{\bar{\epsilon}_2}\right)\frac{\partial F_{2jk}}{\partial X_k}\right.$$
$$\left.+C_{P4}s^2F_{2jk}\frac{\partial^2\bar{\epsilon}_2}{\partial X_k\partial X_l}C\left(\frac{\overline{m_{1l}}}{\bar{\epsilon}_1}-\frac{\overline{m_{2l}}}{\bar{\epsilon}_2}\right)\right]-4\frac{\psi}{s}(\beta_{21}F_{1kk}+\beta_{22}F_{2kk})^{1/2}F_{2ij} \qquad [A.2]$$

and

$$\frac{\partial T_{ij}}{\partial t}+T_{ij}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1k}}}{\bar{\epsilon}_1}+\frac{\overline{m_{2k}}}{\bar{\epsilon}_2}\right)+T_{kj}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{1i}}}{\bar{\epsilon}_1}\right)+T_{ik}\frac{\partial}{\partial X_k}\left(\frac{\overline{m_{2j}}}{\bar{\epsilon}_2}\right)+\tilde{U}_k\frac{\partial T_{ij}}{\partial X_k}$$
$$=\frac{s}{(\beta_1F_{1kk}+\beta_2F_{2kk})^{1/2}}\frac{\partial}{\partial X_k}\left[F_{2kl}\frac{\partial}{\partial X_l}\left(\frac{F_{1ij}\bar{\epsilon}_2}{\bar{\epsilon}_1}\right)+F_{1il}\frac{\partial}{\partial X_l}\left(\frac{T_{kj}}{\bar{\epsilon}_1}\right)+F_{1jl}\frac{\partial}{\partial X_l}\left(\frac{T_{ik}}{\bar{\epsilon}_1}\right)\right]$$
$$+\frac{s}{(\beta_1F_{1kk}+\beta_2F_{2kk})^{1/2}}\frac{\partial}{\partial X_k}\left[F_{1kl}\frac{\partial}{\partial X_l}\left(\frac{F_{2ij}\bar{\epsilon}_1}{\bar{\epsilon}_2}\right)+F_{2il}\frac{\partial}{\partial X_l}\left(\frac{T_{ki}}{\bar{\epsilon}_2}\right)+F_{2jl}\frac{\partial}{\partial X_l}\left(\frac{T_{ik}}{\bar{\epsilon}_2}\right)\right]$$
$$+C\frac{\bar{\epsilon}_1}{\rho_1}(\bar{\epsilon}_2F_{1ij}-T_{ij})+c\frac{\bar{\epsilon}_2}{\rho_2}(\bar{\epsilon}_1F_{2ij}-T_{ij})+\tfrac{1}{2}k_1\rho_1\frac{\omega}{s}(\beta_1F_{1kk}+F_{2kk})^{1/2}\left(\frac{\delta_{ij}}{3}F_{2kk}-F_{2ij}\right)$$
$$+\frac{1}{2}k_2\rho_2\frac{\omega}{s}(\beta_1F_{1kk}+\beta_2F_{2kk})^{1/2}\left(\frac{\delta_{ij}}{3}F_{2kk}-F_{2ij}\right)-4\frac{\psi}{s}\left(\frac{F_{1kk}}{\bar{\epsilon}_1}+\frac{F_{2kk}}{\bar{\epsilon}_2}-\frac{2U_{kk}}{\bar{\epsilon}_1\bar{\epsilon}_2}\right)^{1/2}T_{ij}. \qquad [A.3]$$